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The fermionic observable and
the inverse Kac-Ward operator

The discrete fermionic observable for the FK-Ising model on the square lattice was introduced by Smirnov in [53] (although, as mentioned in [14], similar objects appeared in earlier works). He proved in [54] that the scaling limit of the observable at criticality is given by the solution to a Riemann–Hilbert boundary value problem, and therefore is conformally covariant. A generalization of this result to Ising models defined on a large class of isoradial graphs was obtained by Chelkak and Smirnov in [14], yielding also universality of the scaling limit.

Since then, several different types of observables have been proposed for both the random cluster and classical spin Ising model. They were used to prove conformal invariance of important quantities in these models. The scaling limit of the energy density of the critical spin Ising model on the square lattice was computed by Hongler and Smirnov [34]. Existence and conformal invariance of the scaling limits of the magnetization and multi-point spin correlations were established by Chelkak, Hongler and Izyurov [13]. The fermionic observables are also among the tools used by Chelkak et al. to prove convergence of the critical Ising interfaces to SLE curves [12]. Moreover, the observables also proved useful in the off-critical regime and were employed by Beffara and Duminil-Copin [6] to give a new proof of criticality of the self-dual point and to calculate the correlation length in the Ising model on the square lattice. In a more recent work of Hongler, Kytölä and Zahabi [33], the fermionic observables were identified as correlation functions of fermion operators in the transfer matrix formalism for the same model. One also has to mention the relation between the fermionic observable and the inverse Kasteleyn operator which was pointed out by Dubédat [22].

In this chapter, we establish a direct connection between the fermionic observable for the spin Ising model and the inverse Kac–Ward operator. In Section 5.1, we describe properties of the complex weights induced by the Kac–Ward operator on the non-backtracking walks in the graph. We then use loop expansions of the even subgraph generating function from Chapter 2 to express the inverse Kac–Ward operator on a finite graph in terms of a weighted sum over a certain family of subgraphs. We call the resulting formula the fermionic generating function since it bears a strong resemblance to the definitions of the spin fermionic observables from [14, 33, 34]. In Section 5.2,

we work on isoradial graphs. First, we consider the Kac–Ward operator correspond-
ing to the critical Ising model and we show that it can be thought of as the operator
of s-holomorphicity. To be precise, we prove that a function is s-holomorphic if and
only if it lies in the kernel of the critical Kac–Ward operator composed with a certain
projection operator. Moreover, the inverse Kac–Ward operator can be identified with
the Green's function for a discrete Riemann–Hilbert boundary value problem similar to
the ones considered in [13, 14, 32, 54]. Subsequently, using bounds from Chapter 4,
we show that in finite volume the inverted critical Kac–Ward operator admits a repre-
sentation in terms of non-backtracking walks, whereas a continuous inverse in infinite
volume does not exist. We also consider the supercritical (high-temperature) inverse
operators. They too are expressed in terms of walks (both on finite and infinite graphs),
and moreover, the associated Green's function decays exponentially fast with the dis-
tance between two edges. In particular, the supercritical operator on the full isoradial
graph has a continuous inverse.

We would like to mention that some of the results of this chapter were obtained
independently by Cimasoni [17] in a more general setting of surface graphs.

As a remark, we would like to point out that our observations seem to fit into a more
general picture of two-dimensional discrete physical models satisfying the following
three conditions:

   (i) the partition function of the model is equal to the square root of the determinant
        of some operator,

   (ii) an important observable in the model is given by the inverse of this operator,

   (iii) the critical values of parameters of the model coincide with the values of parame-
         ters which make this operator into some (massless) discrete differential operator.

Our results show that the Ising model on isoradial graphs satisfies this classification
with the distinguished operator being the Kac–Ward operator, the observable being the
fermionic observable, and the discrete differential operator being the s-holomorphic
operator. Another example is the discrete Gaussian free field, where the partition func-
tion is equal to the square root of the determinant of the discrete Laplacian, and the
two-point spin correlation functions are given by the inverse of the Laplacian. More-
over, the general picture of the non-backtracking walk representation of the inverse
Kac–Ward operator is analogous to the one of the random walk representation of the
inverse Laplacian [10]. This similarity can also be seen between the representations of
the solutions to the discrete Riemann–Hilbert boundary value problem and the discrete
Dirichlet boundary value problem for harmonic functions. Also the dimer model [39],
which is known to be closely related to the Ising model, fits the above pattern. The
square of the partition sum of this model is equal to the absolute value of the determi-
nant of the Kasteleyn operator, which acts as the discrete Dirac operator (see e.g. [40]).
Moreover, the observable of main interest in the work of Kenyon [40] is the coupling
function defined as the inverse of the Kasteleyn operator.

5.1 The inverse Kac–Ward operator and non-backtracking walks

We assume that $\mathcal{G} = (V, E)$ is a (possibly infinite) graph with finite maximum degree, and $x = (x_e)_{e \in E}$ is a vector of positive edge weights satisfying

$$\|x\|_\infty = \sup_{e \in E} x_e < \infty.$$

Let $\vec{E}$ be the set of the directed edges of $\mathcal{G}$. The Kac–Ward operator, as defined in (4.8), is an automorphism of the complex vector space $\mathbb{C}^{\vec{E}}$ defined via matrix multiplication by the matrix

$$T(x) = \mathrm{Id} - \Lambda(x), \tag{5.1}$$

where Id is the identity and Λ is the transition matrix. Note that this is well defined in our setting since T has at most Δ nonzero entries in each row, where Δ is the maximum degree of $\mathcal{G}$. Moreover, since the weight vector is bounded, T is continuous (bounded) when treated as an operator on the Hilbert space $\ell^2(\vec{E})$. We already know from Theorem 4.4 that if $\mathcal{G}$ is finite and no two edges of $\mathcal{G}$ cross each other, then the determinant of T is proportional to the square of the partition function of the corresponding Ising model. In particular, in this case T is an isomorphism.

5.1.1 Signed non-backtracking walks

In Chapter 2, we defined walks as sequences of vertices in the graph. For the purpose of the chapter, it will be more convenient to define walks as sequences of directed edges. Hence, a *(non-backtracking) walk* ω of *length* n in $\mathcal{G}$ is a sequence of directed edges $\omega = (\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_n) \in \vec{E}^{n+1}$, such that $h(\vec{e}_i) = t(\vec{e}_{i+1})$ (recall the notation from Section 4.2) and $\vec{e}_{i+1} \neq -\vec{e}_i$ for $i = 0, \ldots, n-1$. Note that the length counts the number of steps that ω makes between edges, rather than the number of edges it visits. A walk ω is *closed* if $\vec{e}_0 = \vec{e}_n$ and if its length is at least 1. We say that ω *goes through* a directed edge $\vec{e}$ (undirected edge e) if $\vec{e}_i = \vec{e}$ ($e_i = e$), for some $i \in \{0, \ldots, n-1\}$. Note that ω does not necessarily go through $\vec{e}_n$, and in particular, walks of length zero do not go through any edge. A walk is called a *path* if it goes through every undirected edge at most once. By ω^{-1} we mean the reversed walk $(-\vec{e}_n, -\vec{e}_{n-1}, \ldots, -\vec{e}_0)$.

The *(signed) weight* of a walk $\omega = (\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_n)$ is given by

$$w(\omega) = e^{\frac{i}{2}\alpha(\omega)} \prod_{i=0}^{n-1} x_{e_i}, \qquad \text{where} \qquad \alpha(\omega) = \sum_{i=0}^{n-1} \angle(\vec{e}_i, \vec{e}_{i+1}) \tag{5.2}$$

is the total turning angle of ω. Note that with this definition of the signed weight, the last edge of ω is not counted in terms of edge weights but does contribute to the total winding angle of ω. If $|\omega| = 0$, then we put $\alpha(\omega) = 0$ and $w(\omega) = 1$. The fundamental

feature of the signed weight is that it factorizes over the steps that a path makes, where the step weight is given by the transition matrix (4.7), i.e.

$$w(\omega) = \prod_{i=0}^{n-1} \Lambda_{\vec{e}_i, \vec{e}_{i+1}}. \tag{5.3}$$

Note that the signed weight of a loop which was defined in Chapter 2 has an opposite sign compared to (5.2). The main reason for this change in notation is the above factorization property of the weight of a walk.

Given two directed edges $\vec{e}$ and $\vec{g}$, we write $\mathcal{W}(\vec{e}, \vec{g})$ for the collection of all walks in $\mathcal{G}$ which start at $\vec{e}$ and end at $\vec{g}$. Since the complex argument satisfies the logarithmic identity $\mathrm{Arg}(z/w) = \mathrm{Arg}(z) - \mathrm{Arg}(w) \pmod{2\pi}$, we conclude that

$$w(\omega) \in e^{\frac{i}{2} \angle(\vec{e}, \vec{g})} \mathbb{R} \qquad \text{for } \omega \in \mathcal{W}(\vec{e}, \vec{g}). \tag{5.4}$$

On the other hand, since walks are non-backtracking and $\mathrm{Arg}(1/z) = -\mathrm{Arg}(z)$ for $z \notin (-\infty, 0]$, it follows that $\alpha(\omega) = -\alpha(\omega^{-1})$. Combining these two facts, we obtain that

$$w(\omega) = \begin{cases} -w(\omega^{-1}) & \text{if } \omega \in \mathcal{W}(\vec{e}, -\vec{e}); \\ w(\omega^{-1}) & \text{if } \omega \in \mathcal{W}(\vec{e}, \vec{e}). \end{cases} \tag{5.5}$$

The first identity in (5.5) implies cancellations of weights of walks which go through certain edges in both directions. The most basic consequence of this property is the following lemma:

Lemma 5.1. *For any $\vec{e} \in \vec{E}$,*

$$\sum_{\omega \in \mathcal{W}(\vec{e}, -\vec{e})} w(\omega) = 0.$$

Proof. If $\mathcal{W}(\vec{e}, -\vec{e})$ is empty then the above statement is trivially true. Otherwise, if $\omega \in \mathcal{W}(\vec{e}, -\vec{e})$, then $\omega^{-1} \in \mathcal{W}(\vec{e}, -\vec{e})$, $w(\omega) = -w(\omega^{-1})$, and $(\omega^{-1})^{-1} = \omega$. Hence, we have cancellation of all terms in the series. $\square$

This observation, and others which naturally follow from property (5.5) (see Lemma 5.7 and 5.8 in Section 5.3) will be important in the computation of the inverse of the Kac–Ward operator.

Note that the above sum is, in general, an infinite power series in the variables x_e. To be rigorous when dealing with power series, we will always assume, unless stated otherwise, that $\|x\|_\infty$ is sufficiently small for the series to be absolutely convergent. In all of the cases, it will be enough to take $\|x\|_\infty < (\Delta - 1)^{-1}$ as in Theorem 2.9.

5.1.2 The inverse Kac–Ward operator

In this section, we assume that $\mathcal{G} = (V, E)$ is finite and without edge crossings. We already know that in this case, the inverse Kac–Ward operator exists. If one wants to compute it, one can use the power series formula:

$$T^{-1}_{\vec{e},\vec{g}} = (\text{Id} - \Lambda)^{-1}_{\vec{e},\vec{g}} = \sum_{n=0}^{\infty} \Lambda^n_{\vec{e},\vec{g}} = \sum_{\omega \in \mathcal{W}(\vec{e},\vec{g})} w(\omega), \qquad (5.6)$$

which is valid for $\|x\|_\infty$ small enough. The last sum is over the non-backtracking walks only since the transition matrix Λ assigns zero weight to steps between $\vec{e}$ and $-\vec{e}$. It turns out that this sum can be expressed in terms of a generating function of certain subgraphs of $\mathcal{G}$ (or rather its particular modification).

To this end, let $m(\vec{e}) = (t(\vec{e}) + h(\vec{e}))/2$ be the midpoint of $\vec{e}$. Given $\vec{e}, \vec{g} \in \vec{E}$, we define a modified graph $\mathcal{G}_{\vec{e},\vec{g}} = (V_{\vec{e},\vec{g}}, E_{\vec{e},\vec{g}})$, which instead of e and g, contains appropriate *half-edges*, i.e., where

$$V_{\vec{e},\vec{g}} = V \cup \{m(\vec{e}), m(\vec{g})\} \quad \text{and} \quad E_{\vec{e},\vec{g}} = \big(E \backslash \{e, g\}\big) \cup \big\{\{m(\vec{e}), h(\vec{e})\}, \{t(\vec{g}), m(\vec{g})\}\big\}$$

We also modify the weight vector x by setting the weight of $\{m(\vec{e}), h(\vec{e})\}$ to be x_e, and in the case when $\vec{g} \neq -\vec{e}$, the weight of $\{t(\vec{g}), m(\vec{g})\}$ to be 1. We write $\mathcal{E}_{\vec{e},\vec{g}}$ for the collection of subsets $H \subset E_{\vec{e},\vec{g}}$ containing the half-edges $\{m(\vec{e}), h(\vec{e})\}$ and $\{t(\vec{g}), m(\vec{g})\}$, and such that all vertices from V have even degree in the graph $(V_{\vec{e},\vec{g}}, H)$ (see Figure 5.1). Note that we do not require that $m(\vec{e})$ and $m(\vec{g})$ have even degree. It follows that $\mathcal{E}_{\vec{e},-\vec{e}}$ is empty, since there is no graph which has exactly one vertex with odd degree. Also note that $\mathcal{E}_{\vec{e},\vec{e}}$ is in bijective correspondence with the set of even subgraphs of $\mathcal{G}$ containing e.

Suppose that $\vec{g} \neq -\vec{e}$ and take $H \in \mathcal{E}_{\vec{e},\vec{g}}$. It follows that there is a path in H which starts at $(m(\vec{e}), h(\vec{e}))$ and ends at $(t(\vec{g}), m(\vec{g}))$. Let ω_H be the left-most such path, i.e. the path which always makes a step to the left-most edge which has not yet been visited in any direction. Note that H splits into ω_H and an even subgraph of $\mathcal{G}$ (see Figure 5.1). Since H also belongs to $\mathcal{E}_{-\vec{g},-\vec{e}}$ this notation may be ambiguous (the reversed left-most path becomes the right-most path), but we will always use it in the context of fixed edges $\vec{e}$ and $\vec{g}$.

For $\vec{e}, \vec{g} \in \vec{E}$, we define the *fermionic generating function* by

$$F_{\vec{e},\vec{g}} = \delta_{\vec{e},\vec{g}} + \frac{1}{Z} \sum_{H \in \mathcal{E}_{\vec{e},\vec{g}}} e^{-\frac{i}{2}\alpha(\omega_H)} \prod_{h \in H} x_h, \qquad (5.7)$$

where δ is the Kronecker delta and $Z = Z(x)$ is the even subgraph generating function (2.10) defined for $\mathcal{G}$. For $\vec{g} = -\vec{e}$, the above sum is empty and we take it to be zero. Note the resemblance between this definition and the definitions of fermionic observables from [14, 33, 34]. The difference is that the fermionic generating function is a function of two directed edges and the fermionic observable from the literature can
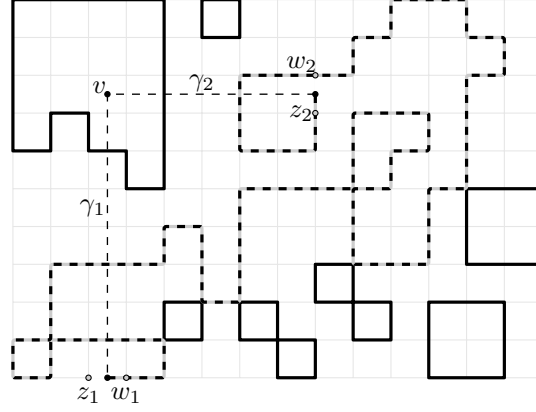
Figure 5.1: In this case, $\mathcal{G}$ is a rectangular piece of the square lattice. A graph $H \in \mathcal{E}_{\vec{e},\vec{g}}$ is drawn in bold lines, where $\vec{e} = (z_1, w_1)$ and $\vec{g} = (z_2, w_2)$. The graph splits into two parts: the path ω_H represented by the dashed lines, and an even subgraph of $\mathcal{G}$. Adding two edges γ_1 and γ_2 makes ω_H into a closed path with three self-crossings, and H into an even subgraph of $E_{\vec{e},\vec{g}} \cup \{\gamma_1, \gamma_2\}$ with five edge crossings (see the proof of Theorem 5.2).

be seen as a function of one directed and one undirected edge. Indeed, for regular lattices (the square, triangular and hexagonal lattice), the fermionic observable is, up to a complex multiplicative constant, the symmetrization in the variable $\vec{g}$ of the fermionic generating function, i.e. the sum over the two opposite orientations of the undirected edge g. For general isoradial graphs it becomes a weighted symmetrization, where the weight depends on the local geometry of the graph (see Section 5.2).

Recall that we assume that $\mathcal{G}$ is finite and does not have any edge crossings. We can now state the main theorem of this section:

Theorem 5.2. *For any $\vec{e}, \vec{g} \in \vec{E}$,*

$$\overline{F_{\vec{e},\vec{g}}} = \sum_{\omega \in \mathcal{W}(\vec{e},\vec{g})} w(\omega).$$

For the proof of this result, see Section 5.3.1. As a direct corollary, we get that $\overline{F} = \left(\overline{F_{\vec{e},\vec{g}}}\right)_{\vec{e},\vec{g} \in \vec{E}}$ is the inverse Kac–Ward operator:

Corollary 5.3. *The inverse Kac–Ward operator on a finite graph $\mathcal{G}$ with no edge crossings is the complex conjugate of the fermionic generating function, i.e.*

$$T^{-1} = \overline{F}.$$

Proof. Fix $\vec{e}, \vec{g} \in \vec{E}$ and x. Consider the rescaled weight vector tx, where t is a positive real number. Since $\mathcal{G}$ has no edge crossings, Z is never zero by (2.10) and it

follows from Theorem 4.4 that $\det T$ is also never zero. Hence, $\overline{F_{\vec{e},\vec{g}}}$ and $T^{-1}_{\vec{e},\vec{g}}$, treated as functions of the scaling factor t, are analytic on $(0, \infty)$. By uniqueness of the analytic continuation, it is enough to prove the desired equality for t small, and this follows from Theorem 5.2 and the power series expansion (5.6). $\qquad\square$

Note that the fermionic generating function was defined only for finite graphs. Theorem 5.2 and Corollary 5.3 give two interpretations of F which do not require finiteness of the underlying graph. We will discuss this issue in Section 5.2.

5.2 The Kac–Ward operator on isoradial graphs

In this section, we assume that $\mathcal{G} = (V, E)$ is a subgraph of an infinite isoradial graph $\Gamma = (V_\Gamma, E_\Gamma)$. Recall from Chapter 4 that this means that all faces of Γ can be inscribed into circles with a common radius and the circumcenters lie within the corresponding faces. Equivalently, the dual graph Γ^* can be embedded in such a way that all pairs of mutually dual edges e and e^* form diagonals of rhombi. For each edge e, let θ_e be the undirected angle between e and any side of the rhombus associated to e (see Figure 4.2). We will consider a family of weight vectors given by

$$x_e(\beta) = \tanh \beta J_e, \qquad \text{where} \qquad \tanh J_e = \tan(\theta_e/2), \qquad (5.8)$$

and where $\beta \in (0, 1]$ is the inverse temperature and $e \in E_\Gamma$. These weights come from the high-temperature expansion of the Ising model and the numbers J_e are called the coupling constants (see Chapter 4). In the case when $\beta = 1$, we will talk about the *critical* weight vector and for $\beta \in (0, 1)$, the weights will be called *supercritical*. The critical case corresponds to the self-dual Z-invariant Ising model which was introduced by Baxter [2].

5.2.1 The critical Kac–Ward operator and s-holomorphicity

In this section, we assume that the weight vector is critical. The notion of *s-holomorphicity* (s stands for *strong* or *spin*) was introduced in [54] in the setting of the square lattice, and was later generalized in [14] to fit the context of general isoradial graphs. Our definition of s-holomorphicity will be equivalent to that in [14], up to multiplication of the function by some globally fixed complex constant.

Consider a vertex z in Γ and let z^* be a vertex in Γ^* corresponding to one of the faces of Γ incident to z. By e_1 and e_2 we denote the two edges lying on the boundary of this face and having z as an endpoint (see Figure 4.2). We say that a complex function f defined on the edges of Γ is *s-holomorphic* at z if for all such dual vertices z^* and the corresponding edges e_1 and e_2,

$$\text{Proj}(f(e_1); (z - z^*)^{-\frac{1}{2}}\mathbb{R}) = \text{Proj}(f(e_2); (z - z^*)^{-\frac{1}{2}}\mathbb{R}),$$

where $\mathrm{Proj}(w; l)$ is the orthogonal projection of the complex number w onto the line l. Note that the choice of the square root is immaterial in the definition above. The property of being s-holomorphic is a real linear property, i.e. addition of two functions and multiplication of a function by a real number preserves s-holomorphicity. It is also a stronger property than the usual discrete holomorphicity: if a function is s-holomorphic at z, then the same function considered as a function on the dual edges is discrete holomorphic at z, i.e. the discrete contour integral around the face corresponding to z vanishes. On the other hand, each discrete holomorphic function is, up to an additive constant, uniquely represented as a sum of two s-holomorphic functions, where one of them is multiplied by i. For proofs of these facts and other properties of s-holomorphic functions, see [14].

The Kac–Ward operator for $\mathcal{G}$ was defined in Section 5.1 as an automorphism of the complex vector space $\mathbb{C}^{\vec{E}}$ but it can also be seen as an operator acting on a smaller real vector space. To be precise, to each directed edge $\vec{e}$ we associate a line $l_{\vec{e}}$ in the complex plane defined by

$$l_{\vec{e}} = e^{-\frac{i}{2}\angle(\vec{e})}\mathbb{R}, \qquad \text{where} \quad \angle(\vec{e}) = \mathrm{Arg}(h(\vec{e}) - t(\vec{e})).$$

As before, we use the principal value of the complex argument. Note that $l_{\vec{e}}$ and $l_{-\vec{e}}$ are orthogonal and they can be thought of as "a local coordinate system at e". We will consider the direct product of the lines treated as one-dimensional real vector spaces, i.e. we put

$$\mathcal{L} = \prod_{\vec{e} \in \vec{E}} l_{\vec{e}}.$$

By the logarithmic property of the complex argument, $T_{\vec{e},\vec{g}}$ defines by multiplication a linear map from $l_{\vec{g}}$ to $l_{\vec{e}}$. This means that the Kac–Ward operator can be seen as an automorphism of $\mathcal{L}$. We define $\mathcal{Y}$ to be $\mathbb{C}^{\mathcal{G}}$ treated as a real vector space and we consider an isomorphism between $\mathcal{Y}$ and $\mathcal{L}$ given by

$$Sf(\vec{e}) = \sin(\theta_e/2)\mathrm{Proj}(f(e); l_{\vec{e}}) \qquad \text{for } f \in \mathcal{Y}.$$

If Γ is a regular lattice (the square, triangular or hexagonal lattice), then all the angles θ_e are equal and S is proportional to the projection operator which gives "local coordinates" at each edge. Note that the inverse of S is given by

$$S^{-1}\varphi(e) = \left(\sin(\theta_e/2)\right)^{-1}\left(\varphi(\vec{e}) + \varphi(-\vec{e})\right) \qquad \text{for } \varphi \in \mathcal{L}.$$

We say that z is an interior vertex of $\mathcal{G}$ if the degrees of z in $\mathcal{G}$ and Γ are the same. Recall from Chapter 4 that the set of directed edges emanating from a vertex z is denoted by $\mathrm{Out}(z)$. We will write $\mathrm{In}(z) = \{\vec{e} \in \vec{E} : h(\vec{e}) = z\} = -\mathrm{Out}(z)$ for the set of edges pointing at z. The next result expresses the fact that the critical Kac–Ward operator (composed with S) can be seen as the operator of s-holomorphicity:

Theorem 5.4. *Let T be the critical Kac–Ward operator. A function $f \in \mathcal{Y}$ is s-holomorphic at an interior vertex z if and only if*

$$TSf(\vec{e}) = 0 \qquad \text{for all } \vec{e} \in \text{In}(z).$$

The proof of this theorem is given in Section 5.3.2.

Consider the case where $\mathcal{G}$ is the full Γ and take f to be equal to 1 everywhere. Of course, f is s-holomorphic at all vertices of Γ. It follows from the theorem above that TSf is equal to zero everywhere and hence the critical Kac–Ward operator for the full isoradial graph has a nontrivial kernel. Hence, it is not invertible on $\mathcal{L}$ and therefore also on $\mathbb{C}^{\vec{E}}$.

Let us go back to the case where $\mathcal{G}$ is a finite subgraph of Γ. From Section 5.1.2, we know that the inverse Kac–Ward operator exists for all weight vectors on $\mathcal{G}$. As a consequence of Theorem 5.4, we can construct s-holomorphic functions by applying the inverse of TS to functions which are zero almost everywhere. To this end, we define the standard basis of $\mathcal{L}$ to be the set of functions $\{i_{\vec{e}}\}_{\vec{e} \in \vec{E}}$, where $i_{\vec{e}}(\vec{g}) = e^{-\frac{i}{2}\angle(\vec{e})}\delta_{\vec{e},\vec{g}}$. It follows that $f_{\vec{e}} = (TS)^{-1}i_{-\vec{e}}$ is s-holomorphic at all interior vertices of $\mathcal{G}$ which are not $t(\vec{e})$, and is not s-holomorphic at $t(\vec{e})$. We also have that

$$
\begin{aligned}
f_{\vec{e}}(g) &= S^{-1}T^{-1}i_{-\vec{e}}(g) \\
&\propto \left(\sin(\theta_g/2)\right)^{-1}\left(T^{-1}_{\vec{g},-\vec{e}} + T^{-1}_{-\vec{g},-\vec{e}}\right) \\
&\propto \left(\cos(\theta_g/2)\right)^{-1}\left(F_{\vec{e},\vec{g}} + F_{\vec{e},-\vec{g}}\right),
\end{aligned}
$$

where $\propto$ means equality up to a multiplicative constant depending only on $\vec{e}$. We used here Corollary 5.3, the fact that $x_e T^{-1}_{\vec{g},\vec{e}} = x_g \overline{T^{-1}_{-\vec{e},-\vec{g}}}$, and the definition of the critical weight vector. As mentioned before, the cosine term vanishes from this expression if Γ is a regular lattice. Recalling the definition of F, one can see that $f_{\vec{e}}$ is proportional to the critical fermionic observable used in [14, 33, 34].

5.2.2 A discrete Riemann–Hilbert boundary value problem

Let $\mathcal{G} = (V, E)$ be a finite subgraph of Γ induced by the vertex set V. Consider the edges of Γ whose one endpoint belongs to V and the other one to $V_\Gamma \setminus V$. Each such edge splits into two half-edges, one of which is incident to V. We add the incident half-edges to the edge set E and call the resulting set $\overline{E}$. We also add their endpoints (which are the midpoints of the corresponding edges of Γ) to the vertex set V and we call the resulting set $\overline{V}$. Let $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$. Note that since $\mathcal{G}$ is an induced subgraph, all vertices from V are interior in $\overline{\mathcal{G}}$. Let $\vec{n}(\partial\mathcal{G})$ be the set of the directed versions of the half-edges in $\overline{E}$ which point outside $\mathcal{G}$. One can think of them as the "discrete outer normal vectors" to the boundary of $\mathcal{G}$.

Let φ be a function defined on the directed edges of $\overline{\mathcal{G}}$ and satisfying

$$\varphi(\vec{e}) \in l_{\vec{e}} \quad \text{for } \vec{e} \in \vec{n}(\partial\mathcal{G}) \qquad \text{and} \qquad \varphi(\vec{e}) = 0 \quad \text{otherwise.}$$

We say that $f : \overline{E} \to \mathbb{C}$ solves the *discrete Riemann–Hilbert boundary value problem* for the pair $(\mathcal{G}, \varphi)$ if f is s-holomorphic at all $v \in V$ and

$$Sf(\vec{e}) = \varphi(\vec{e}) \qquad \text{for all } \vec{e} \in \vec{n}(\mathcal{G}).$$

Corollary 5.5. *Let $\mathcal{G}$ and φ be as above and let T be the critical Kac–Ward operator defined for $\overline{\mathcal{G}}$, where the half-edges in $\overline{E}$ inherit the weights from the corresponding edges of Γ. Then, the discrete Riemann–Hilbert boundary value problem for $(\mathcal{G}, \varphi)$ has exactly one solution, which is given by*

$$f = S^{-1}T^{-1}\varphi.$$

Proof. Suppose that f is a solution to the discrete Riemann–Hilbert boundary value problem. Note that $\overline{\mathcal{G}}$ is formally not a subgraph of Γ since it contains half-edges of Γ. However, these half-edges are parallel to the corresponding edges of Γ, and therefore we can use Theorem 5.4 to conclude that $TSf(\vec{e}) = 0$ for all $\vec{e} \notin \vec{n}(\mathcal{G})$. Moreover, if $\vec{e} \in \vec{n}(\mathcal{G})$, then $h(\vec{e}) \neq t(\vec{g})$ for all $\vec{g} \neq -\vec{e}$. Hence by (5.1), $TSf(\vec{e}) = \text{Id}Sf(\vec{e}) = \varphi(\vec{e})$ for $\vec{e} \in \vec{n}(\mathcal{G})$. This means that $TSf(\vec{e}) = \varphi(\vec{e})$ for all directed edges $\vec{e}$, and the claim of the lemma follows. $\square$

Note that similar discrete Riemann–Hilbert boundary value problems were introduced in [13, 14, 32, 34, 53], where they were analyzed with the use of the theory of discrete harmonic and holomorphic functions. Moreover, it was proved that the solutions of the discrete problems converge to the solutions of the corresponding continuous Riemann–Hilbert boundary value problems.

5.2.3   The non-backtracking walk representation

In this section, we provide a representation of the inverse Kac–Ward operator in terms of non-backtracking walks. Note that we already used this idea in (5.6) but only for weights which were sufficiently small in the supremum norm. It turns out that the walk expansions on isoradial graphs are valid for both supercritical and critical weight vectors, though their behavior is different in each of these cases.

We will use tools from Chapter 4 and hence we need a regularity condition on Γ, i.e. we will assume that there exist constants k and K such that

$$0 < k \leq \theta_e \leq K < \pi \qquad \text{for all } e \in E_\Gamma. \tag{5.9}$$

Geometrically, this means that the area of the underlying rhombi is uniformly bounded away from zero, or in other words, the rhombi do not get arbitrarily thin.

For $\vec{e}, \vec{g} \in \vec{E}$, we define $\mathcal{W}_n(\vec{e}, \vec{g}) \subset \mathcal{W}(\vec{e}, \vec{g})$ to be the subcollection of all walks of length n, and let $d(\vec{e}, \vec{g})$ be the distance between $\vec{e}$ and $\vec{g}$, i.e. the length of a shortest walk in $\mathcal{W}(\vec{e}, \vec{g})$. All operators in the following statement are treated as operators on the Hilbert space $\ell^2(\vec{E})$.

Theorem 5.6. *If the weights are supercritical and $\mathcal{G}$ is a subgraph of Γ, or the weights are critical and $\mathcal{G}$ is a finite subgraph of Γ, then the inverse Kac–Ward operator is continuous and is given by the matrix*

$$T_{\vec{e},\vec{g}}^{-1} = \sum_{n=d(\vec{e},\vec{g})}^{\infty} \sum_{\omega \in \mathcal{W}_n(\vec{e},\vec{g})} w(\omega).$$

Moreover, in the supercritical case, there exist constants C and $\epsilon < 1$ such that

$$\left| \sum_{\omega \in \mathcal{W}_n(\vec{e},\vec{g})} w(\omega) \right| \leq C\epsilon^n \qquad \text{for all } \vec{e}, \vec{g} \text{ and } n.$$

Furthermore, C and ϵ depend only on β and on the isoradial graph Γ, and do not depend on the particular choice of $\mathcal{G}$. Finally, if $\mathcal{G}$ is the full Γ, then the critical Kac–Ward operator does not have a continuous inverse.

Section 5.3.3 is devoted to the proof of this result. Note that this theorem together with Corollary 5.3 provide a natural definition of the supercritical fermionic generating function on infinite isoradial graphs. Furthermore, the critical fermionic observable on finite graphs also admits a representation in terms of non-backtracking walks and by Corollary 5.5, so does the solution to the discrete Riemann–Hilbert boundary value problem.

As pointed out in the introduction, the picture that Theorem 5.6 presents matches the one of the random walk representation of the inverse Laplacian [10] on the square lattice. Indeed, the inverse of the Laplacian in finite volume is given by the random walk Green's function. Off criticality, i.e. when the Laplacian is massive, the Green's function decays exponentially fast with the distance between two vertices. As a result, the inverse of the massive operator in the whole plane exists and is continuous. On the other hand, the full-plane massless Laplacian does not have a bounded inverse.

The crucial difference between these two representations seems to be the fact that the weights of walks induced by the Laplacian are positive, and therefore yield a measure, whereas the Kac–Ward weights for the non-backtracking walks are complex-valued. In particular, in Theorem 5.6, we have to group the walks by length. Otherwise, the series may diverge. On the other hand, this is not an issue in the random walk representation.

5.3 Proofs of the main results

5.3.1 Proof of Theorem 5.2

Cancellations of signed weights

We already stated Lemma 5.1 as the simplest manifestation of the cancellations of signed weights of the non-backtracking walks. For the proof of Theorem 5.2, we will

also need two slightly more difficult consequences of property (5.5). To this end, for $\vec{e}, \vec{g} \in \vec{E}$, let $\mathcal{V}(\vec{e}, \vec{g}) \subset \mathcal{W}(\vec{e}, \vec{g})$ be the collection of walks which go through e exactly once, and if $e \neq g$, do not go through g (recall from Section 5.1.1 what is meant for a walk to go through an edge). Note that $(\vec{e}) \notin \mathcal{V}(\vec{e}, \vec{e})$. Also, let $\mathcal{U}(\vec{e}, \vec{g}) \subset \mathcal{W}(\vec{e}, \vec{g})$ be the collection of walks which do not go through $-\vec{e}$ and $-\vec{g}$. Note that $\mathcal{U}(\vec{e}, -\vec{e}) = \emptyset$ and $(\vec{e}) \in \mathcal{U}(\vec{e}, \vec{e})$. When necessary, we will denote the dependence of these collections on the underlying graph $\mathcal{G} = (V, E)$ in the subscripts, e.g. we will write $\mathcal{W}_{\mathcal{G}}(\vec{e}, \vec{g})$. Also, if $e \in E$, then with a slight abuse of notation, we will write $\mathcal{G} \setminus \{e\}$ for the graph $(V, E \setminus \{e\})$.

The first property says that the closed walks, which go through their starting edge in both directions, do not contribute to the total sum of weights.

Lemma 5.7. *For any $\vec{e} \in \vec{E}$,*

$$\sum_{\omega \in \mathcal{W}(\vec{e}, \vec{e})} w(\omega) = \sum_{\omega \in \mathcal{U}(\vec{e}, \vec{e})} w(\omega) = \Big(1 - \sum_{\omega \in \mathcal{V}(\vec{e}, \vec{e})} w(\omega)\Big)^{-1}.$$

Proof. If $\mathcal{A} = \mathcal{W}(\vec{e}, \vec{e}) \setminus \mathcal{U}(\vec{e}, \vec{e})$ is empty, then the first equality holds true. Otherwise, take $\omega = (\vec{e}_0, \dots, \vec{e}_n) \in \mathcal{A}$ and note that ω goes through $-\vec{e}$. Let l be the smallest index such that $\vec{e}_l = -\vec{e}$, and let k be the largest index smaller than l such that $\vec{e}_k = \vec{e}$. We define a map $\omega \mapsto \omega'$ by

$$\omega' = (\vec{e}_0, \dots, \vec{e}_{k-1}, -\vec{e}_l, -\vec{e}_{l-1}, \dots, -\vec{e}_k, \vec{e}_{l+1}, \dots, \vec{e}_n).$$

It follows that $\omega' \in \mathcal{A}$ and $(\omega')' = \omega$. By (5.3) and (5.5), we see that $w(\omega) = -w(\omega')$, and therefore the sum of signed weights over $\mathcal{A}$ is zero. To prove the second equality, observe that $\mathcal{U}(\vec{e}, \vec{e})$ maps bijectively to the space of finite sequences of walks from $\mathcal{V}(\vec{e}, \vec{e})$. Indeed, $(\vec{e})$ corresponds to the empty sequence of walks, and for $\omega = (\vec{e}_0, \dots, \vec{e}_n) \in \mathcal{U}(\vec{e}, \vec{e})$ of positive length, let $0 = l_0 < l_1 < \dots < l_m = n$ be the consecutive times when ω visits $\vec{e}$, i.e. $\vec{e}_{l_i} = \vec{e}$ for $i \in \{0, \dots, m\}$. Note that $\omega_i = (\vec{e}_{l_i}, \dots, \vec{e}_{l_{i+1}}) \in \mathcal{V}(\vec{e}, \vec{e})$ for $i \in \{0, \dots, m-1\}$. It follows from (5.3) that $w(\omega) = \prod_{i=0}^{m-1} w(\omega_i)$. Hence, the sum of weights of all walks from $\mathcal{U}(\vec{e}, \vec{e})$, which split into exactly m walks from $\mathcal{V}(\vec{e}, \vec{e})$, equals the mth power of the sum of weights of all walks from $\mathcal{V}(\vec{e}, \vec{e})$. Using the power series expansion of $(1 - t)^{-1}$, we finish the proof. $\square$

The second observation is that, when counting weights of walks going from $\vec{e}$ to $\vec{g}$, it is enough to look at these walks, which visit e for the last time in the direction of $\vec{e}$, and afterwards visit g for the first time in the direction of $\vec{g}$.

Lemma 5.8. *For any $\vec{e}, \vec{g} \in \vec{E}$ such that $e \neq g$,*

$$\sum_{\omega \in \mathcal{W}_{\mathcal{G}}(\vec{e}, \vec{g})} w(\omega) = \sum_{\omega \in \mathcal{W}_{\mathcal{G}}(\vec{e}, \vec{e})} w(\omega) \sum_{\omega \in \mathcal{V}_{\mathcal{G}}(\vec{e}, \vec{g})} w(\omega) \sum_{\omega \in \mathcal{W}_{\mathcal{G} \setminus \{e\}}(\vec{g}, \vec{g})} w(\omega).$$

Proof. Again, if $\mathcal{W}_\mathcal{G}(\vec{e}, \vec{g})$ is empty, then $\mathcal{V}_\mathcal{G}(\vec{e}, \vec{g})$ is also empty and the statement is true. Otherwise, for each $\omega = (\vec{e}_0, \ldots, \vec{e}_n) \in \mathcal{W}_\mathcal{G}(\vec{e}, \vec{g})$, let k be the largest index such that $e_k = e$, and let l be the smallest index larger than k such that $e_l = g$. We define $\omega_{ee} = (\vec{e}_0, \ldots, \vec{e}_k)$, $\omega_{eg} = (\vec{e}_k, \ldots, \vec{e}_l)$ and $\omega_{gg} = (\vec{e}_l, \ldots, \vec{e}_n)$. By (5.3), we have that $w(\omega) = w(\omega_{ee})w(\omega_{eg})w(\omega_{gg})$. It follows from Lemma 5.1 that the contribution of the walks ω such that $\omega_{ee} \in \mathcal{W}_\mathcal{G}(\vec{e}, -\vec{e})$ to the sum on the left-hand side of the desired equality is zero. The same holds for the walks ω with $\omega_{gg} \in \mathcal{W}_{\mathcal{G}\setminus\{e\}}(-\vec{g}, \vec{g})$. Therefore, the only walks ω that contribute to the sum satisfy $\omega_{ee} \in \mathcal{W}_\mathcal{G}(\vec{e}, \vec{e})$, $\omega_{eg} \in \mathcal{V}_\mathcal{G}(\vec{e}, \vec{g})$ and $\omega_{gg} \in \mathcal{W}_{\mathcal{G}\setminus\{e\}}(\vec{g}, \vec{g})$. $\qquad\square$

Note that $\mathcal{V}_\mathcal{G}(\vec{e}, \vec{g})$ may be empty even when $\mathcal{W}_\mathcal{G}(\vec{e}, \vec{g})$ is nonempty.

Dependence of Z on the graph

The next result expresses a multiplicative relation between the generating functions of even subgraphs of $\mathcal{G}$ and $\mathcal{G} \setminus \{e\}$ for some edge e. We will write $Z_\mathcal{G}$ to express the dependence of Z on the graph $\mathcal{G}$.

Lemma 5.9. *For any* $\vec{e} \in \vec{E}$,

$$Z_\mathcal{G} = \Big(1 - \sum_{\omega \in \mathcal{V}_\mathcal{G}(\vec{e}, \vec{e})} w(\omega)\Big) Z_{\mathcal{G}\setminus\{e\}}.$$

Proof. By (2.10), Z is a sum of monomials in x_e, and therefore

$$Z_\mathcal{G} = Z_{\mathcal{G}\setminus\{e\}} + x_e \frac{\partial}{\partial x_e} Z_\mathcal{G}\Big|_{x_e=0}.$$

To compute the partial derivative of $Z_\mathcal{G}$, we use the exponential formula from Theorem 2.10. To see why we obtain the sum over $\mathcal{V}_\mathcal{G}(\vec{e}, \vec{e})$, note that the only loops that survive the evaluation $x_e = 0$ go through e exactly once and hence have multiplicity 1. Moreover, each walk $\omega \in \mathcal{V}_\mathcal{G}(\vec{e}, \vec{e})$ corresponds to exactly one such loop and the signed weight of the loop as defined in (2.7) is minus the signed weight of the closed walk as defined in (5.2). Since putting $x_e = 0$ is equivalent to removing e from $\mathcal{G}$, we use Theorem 2.10 again to express the exponential as $Z_{\mathcal{G}\setminus\{e\}}$. Note that by (2.10) the partial derivative is actually constant in x_e. We still chose to evaluate it at zero since the fact that it does not depend on x_e is not apparent when differentiating the exponential formula. $\qquad\square$

Note that this lemma is related to Lemma 3.4.

Proof of Theorem 5.2

Proof. Let $\mathcal{G} = (V, E)$. The case $\vec{g} = -\vec{e}$ follows from Lemma 5.1 and the fact that $F_{\vec{e},-\vec{e}} = 0$. Next, suppose that $\vec{g} = \vec{e}$ and take $H \in \mathcal{E}_{\vec{e},\vec{e}}$. As mentioned before, H can be thought of as an even subgraph of $\mathcal{G}$ containing e. It follows that the left-most

path ω_H goes along the boundary of the (possibly unbounded) face of H which lies on the left-hand side of $\vec{e}$. This means that ω_H does not have any self-crossings and therefore, by Remark 2.2, $e^{\frac{i}{2}\alpha(\omega_H)} = -1$. Hence by (2.10) and (5.7),

$$\overline{F_{\vec{e},\vec{e}}} = 1 - \frac{1}{Z_{\mathcal{G}}} \sum_{\substack{e \in H \subset E \\ H \text{ even}}} \prod_{g \in H} x_g = \frac{Z_{\mathcal{G}\setminus\{e\}}}{Z_{\mathcal{G}}}.$$

Therefore by Lemma 5.9 and 5.7,

$$\overline{F_{\vec{e},\vec{e}}} = \frac{Z_{\mathcal{G}\setminus\{e\}}}{Z_{\mathcal{G}}} = \left(1 - \sum_{\omega \in \mathcal{V}_{\mathcal{G}}(\vec{e},\vec{e})} w(\omega)\right)^{-1} = \sum_{\omega \in \mathcal{W}_{\mathcal{G}}(\vec{e},\vec{e})} w(\omega). \tag{5.10}$$

The last case is when $e \neq g$. Let $H \in \mathcal{E}_{\vec{e},\vec{g}}$ and $\vec{\gamma} = (m(g), m(e))$. We put $x_\gamma = 1$. Without loss of generality, we assume that γ, when seen as a subset of the plane, does not intersect any vertex from V. This assumption is needed since otherwise, the notions of edge and vertex crossings from Chapter 2 are not well defined. Indeed, if this is not the case, then we can add two edges $\gamma_1 = \{m(e), v\}$ and $\gamma_2 = \{v, m(g)\}$, for some suitably chosen vertex v (see Figure 5.1). The rest of the proof can be easily adjusted to this situation. With a slight abuse of notation, we will write $\mathcal{G}_{\vec{e},\vec{g}} \cup \{\gamma\}$ for the graph $\mathcal{G}_{\vec{e},\vec{g}}$ with γ added to the edge set. Note that $H \cup \{\gamma\}$ is an even subgraph of $\mathcal{G}_{\vec{e},\vec{g}} \cup \{\gamma\}$.

Let ω_H° be the closed path that starts at $\vec{\gamma}$ and then agrees with ω_H until it goes back to $\vec{\gamma}$. We claim that

$$(-1)^{C(H \cup \{\gamma\})} = (-1)^{C(\omega_H^\circ)} = -e^{\frac{i}{2}\alpha(\omega_H^\circ)} = -e^{\frac{i}{2}(\alpha(\omega_H)+\beta)}, \tag{5.11}$$

where $\beta = \angle(\vec{g}, \vec{\gamma}) + \angle(\vec{\gamma}, \vec{e})$, $C(H \cup \{\gamma\})$ is the number of edge crossing in the graph $H \cup \{\gamma\}$, and $C(\omega_H^\circ)$ is the total number of edge and vertex self-crossings of ω_H°. The second equality is a consequence of Remark 2.2, and the last one follows directly from the definitions (4.6) and (5.2). Since H is embedded in the plane without edge crossings, $C(H \cup \{\gamma\})$ is the number of edges in H which are crossed by γ. Similarly, since ω_H always makes a step to the left-most edge, ω_H° does not have any vertex self-crossings. Therefore, $C(\omega_H^\circ)$ is equal to the number of edges in $E(\omega_H)$ which cross γ, where $E(\omega_H)$ is the set of the undirected edges that ω_H visits. What is left to prove, is that the number of edges in $H \setminus E(\omega_H^\circ)$ which are crossed by γ is even. To this end, let $\{\omega_1, \ldots, \omega_k\}$ be a collection of edge-disjoint closed paths, such that $H \setminus E(\omega_H^\circ) = \bigcup_{i=1}^k E(\omega_i)$. Again, since ω_H is the left-most path in H, it is true that ω_H does not have any crossings with ω_i for $i = 1, \ldots, k$. It follows that the total number of crossings between ω_H° and ω_i is equal to the number of edges in $E(\omega_i)$ crossed by γ. Since the total number of crossings between any two edge-disjoint closed paths is even, we have established (5.11).

Note that $H \mapsto H \cup \{\gamma\}$ is a bijection between $\mathcal{E}_{\vec{e},\vec{g}}$ and the collection of even subgraphs of $\mathcal{G}_{\vec{e},\vec{g}} \cup \{\gamma\}$ which contain γ. Similarly to the previous case, from (5.11),

(2.10) and (5.7), it follows that

$$e^{\frac{i}{2}\beta}\overline{F_{\vec{e},\vec{g}}} = \frac{Z_{\mathcal{G}_{\vec{e},\vec{g}}} - Z_{\mathcal{G}_{\vec{e},\vec{g}}\cup\{\gamma\}}}{Z_{\mathcal{G}}} = \left(1 - \frac{Z_{\mathcal{G}_{\vec{e},\vec{g}}\cup\{\gamma\}}}{Z_{\mathcal{G}_{\vec{e},\vec{g}}}}\right)\frac{Z_{\mathcal{G}\setminus\{e,g\}}}{Z_{\mathcal{G}\setminus\{e\}}}\frac{Z_{\mathcal{G}\setminus\{e\}}}{Z_{\mathcal{G}}}, \qquad (5.12)$$

where we also used the fact that $Z_{\mathcal{G}_{\vec{e},\vec{g}}} = Z_{\mathcal{G}\setminus\{e,g\}}$. Using Lemma 5.9, we get

$$1 - \frac{Z_{\mathcal{G}_{\vec{e},\vec{g}}\cup\{\gamma\}}}{Z_{\mathcal{G}_{\vec{e},\vec{g}}}} = \sum_{\omega\in\mathcal{V}_{\mathcal{G}_{\vec{e},\vec{g}}\cup\{\gamma\}}(\vec{\gamma},\vec{\gamma})} w(\omega) = e^{\frac{i}{2}\beta}\sum_{\omega\in\mathcal{V}_{\mathcal{G}}(\vec{e},\vec{g})} w(\omega).$$

Just as in (5.10), the remaining ratios of generating functions in (2.10) can be expressed in terms of walks, and therefore

$$\overline{F_{\vec{e},\vec{g}}} = \sum_{\omega\in\mathcal{V}_{\mathcal{G}}(\vec{e},\vec{g})} w(\omega) \sum_{\omega\in\mathcal{W}_{\mathcal{G}\setminus\{e\}}(\vec{g},\vec{g})} w(\omega) \sum_{\omega\in\mathcal{W}_{\mathcal{G}}(\vec{e},\vec{e})} w(\omega)$$

$$= \sum_{\omega\in\mathcal{W}_{\mathcal{G}}(\vec{e},\vec{g})} w(\omega).$$

The last equality follows from Lemma 5.8. $\qquad\square$

5.3.2 Proof of Theorem 5.4

Proof. We put $\varphi = Sf$. Take two consecutive edges $\vec{e}_1$ and $\vec{e}_2$ from $\text{In}(z)$ ordered counterclockwise around z and suppose that

$$x_{e_1}^{-1}T\varphi(\vec{e}_1) = x_{e_2}^{-1}e^{\frac{i}{2}\angle(\vec{e}_1,\vec{e}_2)}T\varphi(\vec{e}_2). \qquad (5.13)$$

By the definition of T, this is equivalent to

$$\varphi(\vec{e}_1)x_{e_1}^{-1} - e^{\frac{i}{2}\angle(\vec{e}_1,-\vec{e}_2)}\varphi(-\vec{e}_2) - \sum_{\vec{g}\in\text{Out}(z)\setminus\{-\vec{e}_1,-\vec{e}_2\}} \varphi(\vec{g})e^{\frac{i}{2}\angle(\vec{e}_1,\vec{g})} =$$

$$e^{\frac{i}{2}\angle(\vec{e}_1,\vec{e}_2)}\left(\varphi(\vec{e}_2)x_{e_2}^{-1} - e^{\frac{i}{2}\angle(\vec{e}_2,-\vec{e}_1)}\varphi(-\vec{e}_1) - \sum_{\vec{g}\in\text{Out}(z)\setminus\{-\vec{e}_1,-\vec{e}_2\}} \varphi(\vec{g})e^{\frac{i}{2}\angle(\vec{e}_2,\vec{g})}\right).$$

Since the faces of Γ are convex, $\angle(\vec{e}_1,\vec{e}_2) = \theta_{e_1} + \theta_{e_2} > 0$. Using basic properties of the complex argument, one obtains that $\angle(\vec{e}_1,\vec{e}_2) + \angle(\vec{e}_2,-\vec{e}_1) = \pi$, and

$$\angle(\vec{e}_1,\vec{e}_2) + \angle(\vec{e}_2,\vec{g}) = \angle(\vec{e}_1,\vec{g}) \quad \text{for all } \vec{g}\in\text{Out}(z)\setminus\{-\vec{e}_1,-\vec{e}_2\}.$$

Combining this with the equation above, gives

$$\varphi(\vec{e}_1)x_{e_1}^{-1} + i\varphi(-\vec{e}_1) = e^{\frac{i}{2}\angle(\vec{e}_1,\vec{e}_2)}(\varphi(\vec{e}_2)x_{e_2}^{-1} - i\varphi(-\vec{e}_2)), \qquad (5.14)$$

which using criticality of the weights, yields

$$\left(\sin(\theta_{e_1}/2)\right)^{-1}e^{-\frac{i}{2}\theta_{e_1}}\left(\varphi(\vec{e}_1)\cos(\theta_{e_1}/2) + i\varphi(-\vec{e}_1)\sin(\theta_{e_1}/2)\right) = \qquad (5.15)$$

$$\left(\sin(\theta_{e_2}/2)\right)^{-1}e^{\frac{i}{2}\theta_{e_2}}\left(\varphi(\vec{e}_2)\cos(\theta_{e_2}/2) - i\varphi(-\vec{e}_2)\sin(\theta_{e_2}/2)\right).$$

We put

$$l = e^{-\frac{i}{2}\theta_{e_1}} l_{\vec{e}_1} = e^{\frac{i}{2}\theta_{e_2}} l_{\vec{e}_2} = (z - z^*)^{-\frac{1}{2}} \mathbb{R},$$

where z^* is the dual vertex corresponding to the face lying on the right-hand side of $\vec{e}_1$ and $-\vec{e}_2$. From basic geometry it follows that

$$\operatorname{Proj}(z; ze^{i\beta}\mathbb{R}) = ze^{i\beta}\cos\beta \qquad \text{and} \qquad \operatorname{Proj}(z; zie^{i\beta}\mathbb{R}) = -zie^{i\beta}\sin\beta$$

for any nonzero complex number z and any real number β. This, together with the definition of S, the fact that $\varphi(\vec{e}) \in l_{\vec{e}}$ and $l_{\vec{e}} = il_{-\vec{e}}$, implies that equation (5.15) takes the form

$$\operatorname{Proj}(f(e_1); l) = \operatorname{Proj}(f(e_2); l). \tag{5.16}$$

Therefore, condition (5.13) is equivalent to condition (5.16).

Assume that (5.13) holds for all pairs of consecutive edges in $\operatorname{In}(z) = \{\vec{e}_1, \ldots, \vec{e}_k\}$. We obtain that $T\varphi(\vec{e}_1) = e^{\frac{i}{2}\sum_{i=1}^{k} \angle(\vec{e}_i, \vec{e}_{i+1})} T\varphi(\vec{e}_1) = -T\varphi(\vec{e}_1)$, where $\vec{e}_{k+1} = \vec{e}_1$, and hence, $T\varphi(\vec{e}) = 0$ for all $\vec{e} \in \operatorname{In}(z)$. The opposite implication uses the fact that condition (5.13) is trivially satisfied when $T\varphi(\vec{e}_1) = T\varphi(\vec{e}_2) = 0$. $\qquad\square$

5.3.3   Proof of Theorem 5.6

Matrices of operators

If $\mathcal{G}$ is finite, then $\ell^2(\vec{E})$ is a finite dimensional Euclidean space and hence all automorphisms of $\ell^2(\vec{E})$ are continuous and expressed via matrix multiplication. If $\mathcal{G}$ is infinite, then $\ell^2(\vec{E})$ is an infinite dimensional Hilbert space and all its continuous automorphisms are also given by (infinite) matrix multiplication. To be precise, let $\{i_{\vec{e}}\}_{\vec{e}\in\vec{E}}$, where $i_{\vec{e}}(\vec{g}) = \delta_{\vec{e},\vec{g}}$, be the standard basis of $\ell^2(\vec{E})$ and let $\langle \cdot, \cdot \rangle$ be the inner product in $\ell^2(\vec{E})$. If A is a continuous automorphism of $\ell^2(\vec{E})$, then $A_{\vec{e},\vec{g}} := \langle Ai_{\vec{g}}, i_{\vec{e}} \rangle$ are the entries of the associated matrix, and A acts via matrix multiplication, i.e.

$$A\varphi(\vec{e}) = \sum_{\vec{g}\in\vec{E}} A_{\vec{e},\vec{g}}\varphi(\vec{g}) \qquad \text{for all } \varphi \in \ell^2(\vec{E}).$$

The rows of A belong to $\ell^2(\vec{E})$ and hence the order of summation is irrelevant. Moreover, the matrix of a composition of two bounded operators is the product of the two matrices of these operators. Note that the entries of the matrix are given in terms of linear functionals. Hence, whenever a sequence of operators converges in the weak topology, the entries also converge to the entries of the matrix of the limiting operator.

By $\|A\|$ and $\rho(A)$ we will denote the operator norm and the spectral radius of A. From the theory of Banach algebras, we know that

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}, \quad \text{and hence} \quad (\operatorname{Id} - A)^{-1} = \sum_{n=0}^{\infty} A^n \tag{5.17}$$

if $\rho(A) < 1$. Here, Id is the identity on $\ell^2(\vec{E})$ and the limit is taken in the operator norm.

Supercritical case

Proof of exponential decay. Let $\mathcal{G} = (V, E)$ be any subgraph of Γ and fix $\beta \in (0, 1)$. Let the weight vector $x(\beta)$ and the coupling constants J be as in (5.8). By monotonicity of the hyperbolic tangent and by compactness,

$$\epsilon := \sup_{e \in E} \frac{x_e(\beta)}{x_e(1)} \leq \sup_{e \in E_\Gamma} \frac{x_e(\beta)}{x_e(1)} = \sup_{e \in E_\Gamma} \frac{\tanh \beta J_e}{\tanh J_e} \leq \sup_{j \in [m, M]} \frac{\tanh \beta j}{\tanh j} < 1, \quad (5.18)$$

where $\tanh m = \tan(k/2)$ and $\tanh M = \tan(K/2)$, with k and K as in (5.9).

Let D be an isomorphism of $\ell^2(\vec{E})$, which for each directed edge $\vec{e}$ rescales the coordinate corresponding to $\vec{e}$ by $\sqrt{x_e(\beta)}$. Because of condition (5.9), D is bounded and has a bounded inverse D^{-1}. Let $B = D^{-1}\Lambda D$, where Λ is the transition matrix for $\mathcal{G}$ and the weight vector $x(\beta)$. In the language of Chapter 4, B is a conjugated Kac–Ward transition matrix. We will use Corollary 4.8 which explicitly gives the operator norm of a conjugated transition matrix. To this end, note that the angles θ sum up to π around each vertex of Γ (see Figure 4.2). Hence, by (5.8) and (5.18),

$$\sum_{\vec{e} \in \mathrm{Out}(z)} \arctan\big(x_e(\beta)/\epsilon\big) \leq \sum_{\vec{e} \in \mathrm{Out}(z)} \arctan\big(x_e(1)\big) \qquad (5.19)$$

$$= \sum_{\vec{e} \in \mathrm{Out}(z)} \theta_e/2 \leq \pi/2$$

for all vertices z. From the above inequality, Corollary 4.8 and Remark 4.10, it follows that the operator norm of B is bounded from above by ϵ. The operator norm gives an upper bound on the spectral radius, and since B has the same spectrum as Λ, the spectral radius of Λ is not larger than ϵ. To compute the inverse Kac–Ward operator, we can therefore use the power series expansion (5.17) with $A = \Lambda$. To get the non-backtracking walk representation, we compute the powers of Λ using matrix multiplication and we use identity (5.3). We also use the fact that convergence in norm is stronger than weak convergence and hence implies convergence of the entries of the corresponding matrices.

Furthermore, note that for all $\vec{e}$ and $\vec{g}$

$$|\Lambda^n_{\vec{e}, \vec{g}}| = |(DB^n D^{-1})_{\vec{e}, \vec{g}}| = |\langle DB^n D^{-1} i_{\vec{g}}, i_{\vec{e}} \rangle| \leq \|DB^n D^{-1} i_{\vec{g}}\| \cdot \|i_{\vec{e}}\|$$
$$\leq \|DB^n D^{-1}\| \leq \|D\| \cdot \|D^{-1}\| \cdot \|B\|^n \leq C\epsilon^n,$$

where we used the Cauchy-Schwarz inequality and submultiplicativity of the operator norm. Note that both C and ϵ are universal for all subgraphs $\mathcal{G}$, and moreover, (5.18) and (5.9) give explicit upper bounds on these constants. $\square$

Critical case

Proof of the non-backtracking walk expansion. As we already mentioned, the power expansion formula (5.17) is valid whenever the spectral radius of the transition matrix is strictly smaller than one. We will now prove that this is the case if $\mathcal{G} = (V, E)$ is a finite subgraph of Γ and the weight vector is critical.

By Corollary 4.9, for the spectral radius of the critical transition matrix to be strictly smaller than one, it is enough to construct a weight vector $\vec{x}$ such that $\xi_z(\vec{x}) < 1$ for all vertices z, and

$$\vec{x}_{\vec{e}}\vec{x}_{-\vec{e}} = x_e(1) = \tan(\theta_e/2) \qquad \text{for all } \vec{e} \in \vec{E}. \tag{5.20}$$

Let $\partial\mathcal{G} \subset V$ be the set of vertices, whose degree in $\mathcal{G}$ is smaller than in Γ. Note the difference between this definition of the boundary of a graph and the definition from the previous chapter. Moreover, let $\partial_r\mathcal{G} \subset V$ be the set of vertices whose graph distance to $\partial\mathcal{G}$ is at most r. We will inductively construct weight vectors $\vec{x}^r$, which satisfy (5.20), and for which

$$k_z(\vec{x}^r) < 1 \text{ for all } z \in \partial_r\mathcal{G}, \quad \text{and} \quad k_z(\vec{x}^r) = 1 \text{ for all } z \in V \setminus \partial_r\mathcal{G}. \tag{5.21}$$

Indeed, let $\vec{x}_{\vec{e}}^0 = \sqrt{\tan(\theta_e/2)}$. The angles θ sum up to π around each vertex in Γ and hence $\xi_z(\vec{x}^0) = 1$ for all $z \in V \setminus \partial\mathcal{G}$. Since removing an edge incident to a vertex z strictly decreases ξ_z, $\xi_z(\vec{x}^0) < 1$ for all $z \in \partial\mathcal{G}$. Therefore, $\vec{x}^0$ gives the basis of our induction. Now, assume that we already constructed an $\vec{x}^r$ which satisfies (5.20) and (5.21). If $\partial_r\mathcal{G} = V$, then $\vec{x} = \vec{x}^r$ yields the desired bound on the spectral radius. Otherwise, take any $z \in \partial_{r+1}\mathcal{G} \setminus \partial_r\mathcal{G}$ and any $w \in \partial_r\mathcal{G}$ at distance one from z, i.e. such that $\vec{e} = (w, z) \in \vec{E}$. By the induction hypothesis, $\xi_z(\vec{x}^r) = 1$ and $\xi_w(\vec{x}^r) < 1$. By continuity, one can slightly increase $\vec{x}_{\vec{e}}^r$ so that still $\xi_w < 1$. To still satisfy (5.20), the product over the two opposite orientations of e has to remain constant, and hence one has to slightly decrease $\vec{x}_{-\vec{e}}^r$ which results in $\xi_z < 1$. The value of ξ at other vertices does not change. If we do this procedure for all $z \in \partial_{r+1}\mathcal{G}$, it means that we constructed $\vec{x}^{r+1}$ which satisfies (5.20) and (5.21). We proceed until we cover all vertices of $\mathcal{G}$. Note that finiteness of $\mathcal{G}$ is crucial in this reasoning. $\qquad\square$

The full Kac–Ward operator does not have a bounded inverse. Consider the Kac–Ward operator with the critical weight vector on the full isoradial graph $\Gamma = (V_\Gamma, E_\Gamma)$. We already proved that it is not invertible when treated as an operator on the vector space $\mathbb{C}^{\vec{E}_\Gamma}$ since constant functions are in the kernel of TS, where S is the projection operator from Section 5.2.

The idea is similar when T is seen as a continuous operator on $\ell^2(\vec{E})$. We will consider elements of $\ell^2(\vec{E})$ which "approximate" constant functions and show that their images under TS are close to zero. To this end, let $f_\mathcal{G} \in \mathbb{C}^{E_\Gamma}$ be the indicator function of the edge-set of a finite graph $\mathcal{G} = (V, E)$. By the definition of S, $\varphi_\mathcal{G} := Sf_\mathcal{G} \in \ell^2(\vec{E})$ and $\|\varphi_\mathcal{G}\| \geq \sin(k/2)\sqrt{|E|}$, where k is as in (5.9). Note that $f_\mathcal{G}$ is s-holomorphic at all interior vertices of $\mathcal{G}$ and $\Gamma \setminus \mathcal{G}$. By Theorem 5.4, $T\varphi_\mathcal{G}$ can be nonzero only at these directed edges, which point at the vertices of $\partial\mathcal{G}$, where $\partial\mathcal{G}$ is as in the previous proof. From the definition of the Kac–Ward operator, it follows that

$$\|T\varphi_\mathcal{G}\|_\infty \leq \Delta\|x(1)\|_\infty\|\varphi_\mathcal{G}\|_\infty \leq \tan(K/2)\Delta,$$

where Δ is the maximum degree of Γ, and K is as in (5.9). Hence,

$$\|T\varphi_{\mathcal{G}}\| \leq \tan(K/2)\Delta^{3/2}\sqrt{|\partial\mathcal{G}|},$$

which in the end yields

$$\|T\varphi_{\mathcal{G}}\| \leq C\sqrt{|\partial\mathcal{G}|/|E|}\|\varphi_{\mathcal{G}}\|$$

for some constant C independent of $\mathcal{G}$.

It is now enough to notice that Γ admits subgraphs for which the ratio $|\partial\mathcal{G}|/|E|$ is arbitrarily small; it will mean that the inverse, if it exists, is unbounded in norm. To this end, one can consider subgraphs $\mathcal{G}_r$, which are induced by the vertices of Γ contained in the square $[-r, r] \times [-r, r]$. Using condition (5.9), which says that all edges of Γ are surrounded by disjoint rhombi of positive minimal area (and also finite maximal area), one can prove that $|\partial\mathcal{G}_r|$ grows like r and and the number of edges of $\mathcal{G}_r$ grows like r^2 when r goes to infinity. $\qquad\square$